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Basins of Attraction of Cellular Automata and Discrete Dynamical Networks

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Glossary

Attractor, basin of attraction, subtree The terms “attractor” and “basin of attraction” are borrowed from continuous dynamical systems. In this context the attractor signifies the repetitive cycle of states into which the system will settle. The basin of attraction in convergent (injective) dynamics includes the transient states that flow to an attractor as well as the attractor itself, where each state has one successor but possibly zero or more predecessors (pre-images). Convergent dynamics implies a topology of trees rooted on the attractor cycle, though the cycle can have a period of just one, a point attractor. Part of a tree is a subtree defined by its root and number of levels. These mathematical objects may be referred to in general as “attractor basins.”

Basin of attraction field One or more basins of attraction comprising all of state-space.

Cellular automata, CA Although CA are often treated as having infinite size, we are dealing here with finite CA, which usually consist of “cells” arranged in a regular lattice (1D, 2D, 3D) with periodic boundary conditions, making a ring in 1D and a torus in 2D (“null” and other boundary conditions may also apply). Each cell updates its value (usually in parallel, synchronously) as a function of the values of its close local neighbors. Updating across the lattice occurs in discrete time-steps. CA have one homogeneous function, the “rule,” applied to a homogeneous neighborhood template. However, many of these constraints can be relaxed.

Discrete dynamical networks Relaxing RBN constraints by allowing a value range that is greater than binary, $v \geq 2$, heterogeneous k, and a rule-mix.

Garden-of-Eden state A state having no pre-images, also called a leaf state.

Pre-images A state’s immediate predecessors.

Random Boolean Relaxing CA constraints, where each cell can have a different,

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networks, RBN — random (possibly biased) nonlocal neighborhood or put another way random wiring of *k* inputs (but possibly with heterogeneous *k*) and heterogeneous rules (a rule-mix) but possibly just one rule, or a bias of rule types.

Random maps, MAP — Directed graphs with out-degree one, where each state in state-space is *assigned* a successor, possibly at random, or with some bias, or according to a dynamical system. CA, RBN, and DDN, which are usually sparsely connected ($k \ll n$), are all special cases of random maps. Random maps make a basin of attraction field, by definition.

Reverse algorithms — Computer algorithms for generating the pre-images of a network state. The information is applied to generate state transition graphs (attractor basins) according to a graphical convention. The software DDLab, applied here, utilizes three different reverse algorithms. The first two generate pre-images directly so are more efficient than the exhaustive method, allowing greater system size.

- An algorithm for local 1D wiring (Wuensche and Lesser 1992) – 1D CA but rules can be heterogeneous.
- A general algorithm (Wuensche 1994a) for RBN, DDN, and 2D or 3D CA, which also works for the above.
- An exhaustive algorithm that works for any of the above by creating a list of "exhaustive pairs" from forward dynamics. Alternatively, a random list of exhaustive pairs can be created to implement attractor basin of a "random map."

Space-time pattern — A time sequence of states from an initial state driven by the dynamics, making a trajectory. For 1D systems this is usually represented as a succession of horizontal value strings from the top down or scrolling down the screen.

State transition graph — A graph representing attractor basins consisting of directed arcs linking nodes, representing single time-steps linking states, with a topology of trees rooted on attractor cycles, where the direction of time is inward from garden-of-Eden states toward the attractor. Various graphical conventions determine the presentation. The terms "state transition graph" and various types of "attractor basins" may be used interchangeably.

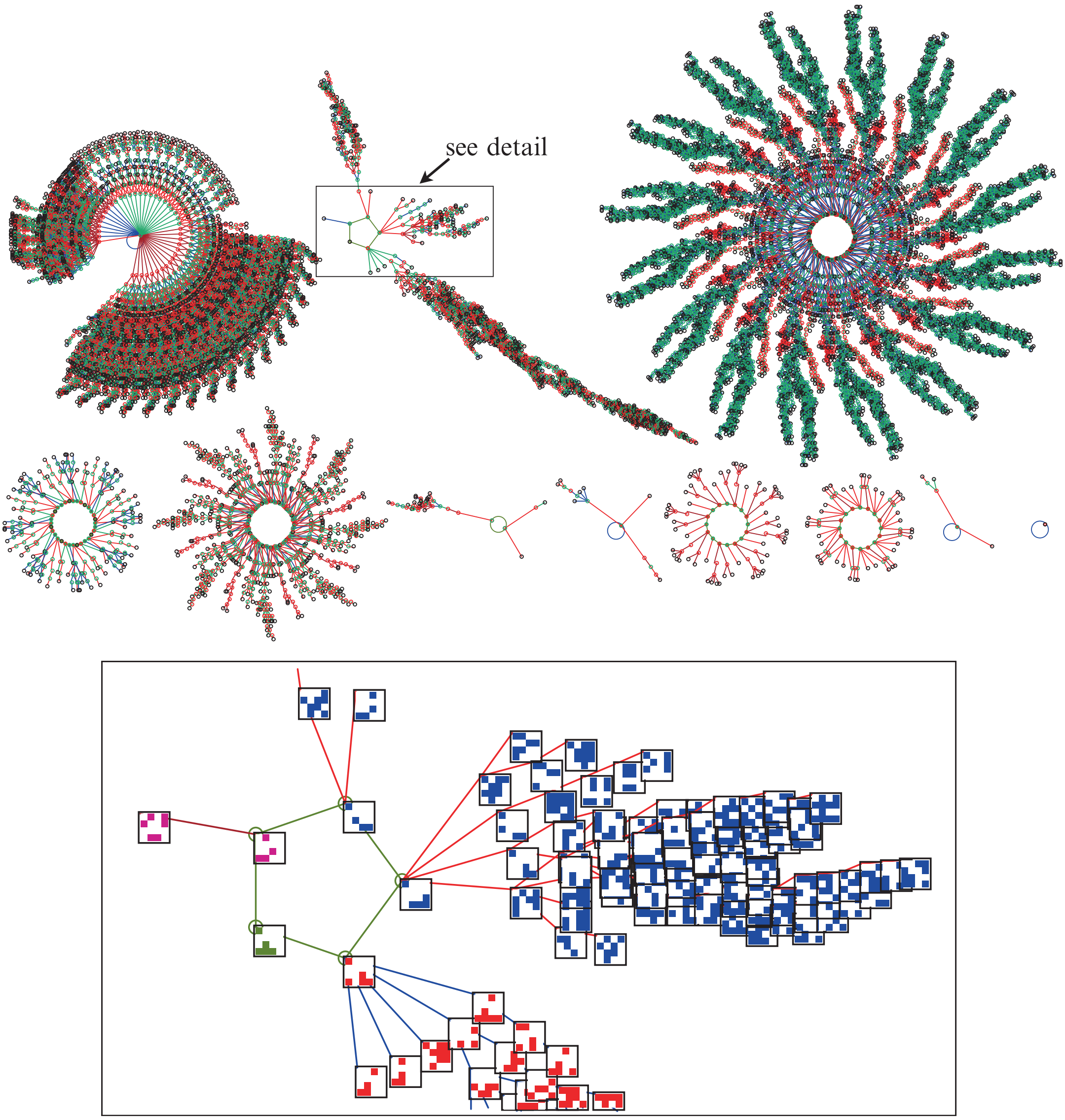
State-space — The set of unique states in a finite and discrete system. For a system of size *n*, and value range *v*, the size of state-space $S = V^n$.

Definition of the Subject

Basins of attraction of cellular automata and discrete dynamical networks link state-space according to deterministic transitions, giving a topology of trees rooted on attractor cycles. Applying reverse algorithms, basins of attraction can be computed and drawn automatically. They provide insights and applications beyond single trajectories, including notions of order, complexity, chaos, self-organization, mutation, the genotype-phenotype, encryption, content-addressable memory, learning, and gene regulation. Attractor basins are interesting as mathematical objects in their own right.

Introduction

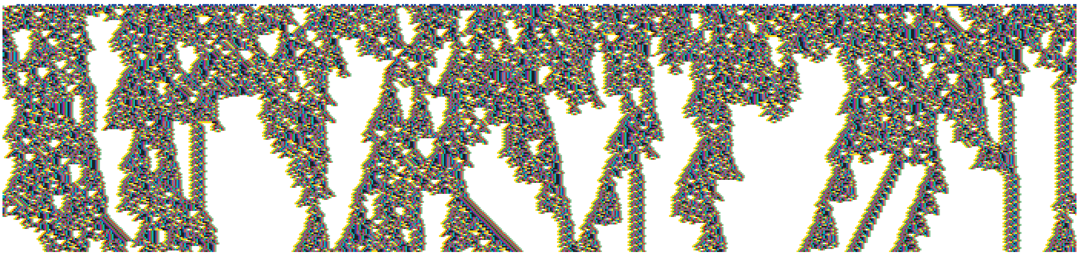
The Global Dynamics of Cellular Automata (Wuensche and Lesser 1992) published in 1992



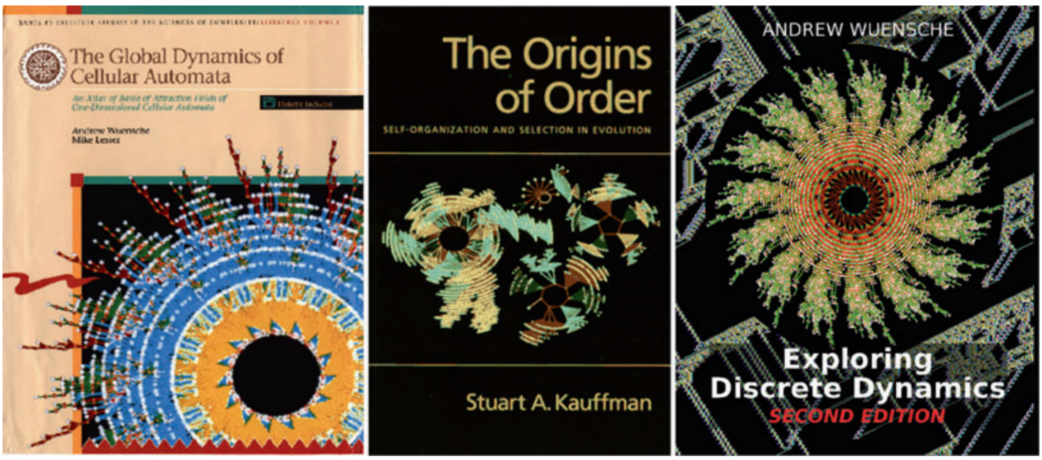
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 1 *Top:* The basin of attraction field of a 1D binary CA, $k = 7$, $n = 16$ (Wuensche 1999). The 2^{16} states in state-space are connected into 89 basins of attraction; only the 11 nonequivalent basins are shown, with symmetries characteristic of CA (Wuensche and Lesser 1992). Time flows inward and then clockwise at the attractor. *Below*: A detail of the second basin, where states are shown as 4×4 bit patterns

introduced a reverse algorithm for computing the pre-images (predecessors) of states for finite 1D binary cellular automata (CA) with periodic boundaries. This made it possible to reveal the precise graph of "basins of attraction" – state transition graphs – states linked into trees rooted on attractor cycles, which could be computed and drawn automatically as in Fig. 1. The book included an atlas for two entire categories of CA rule-space, the three-neighbor "elementary" rules and the five-neighbor totalistic rules (Fig. 2).

In 1993, a different reverse algorithm was invented (Wuensche 1994b) for the pre-images and basins of attraction of random Boolean



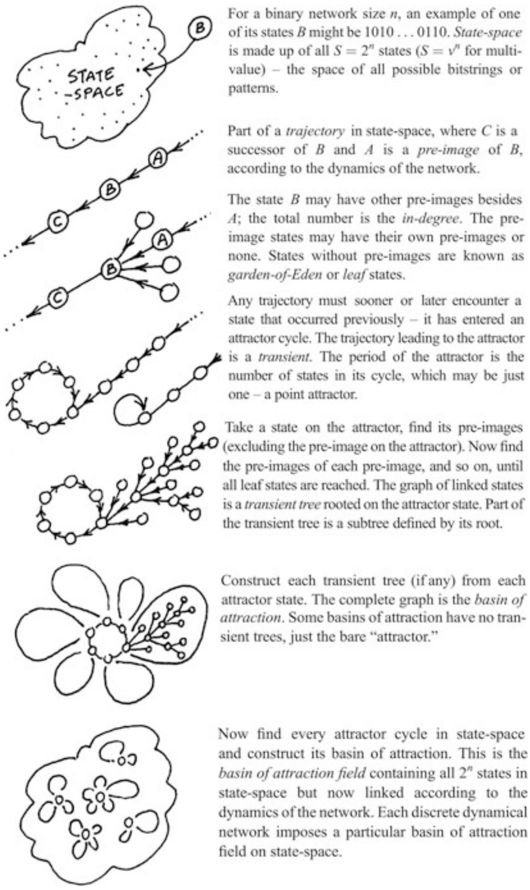
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 2 Space-time pattern for the same CA as in Fig. 1 but for a much larger system ($n = 700$). About 200 time-steps from a random initial state. Space is across and time is down. Cells are colored according to neighborhood lookup instead of the value



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 3 The front covers of Wuensche and Lesser's (1992) *The Global Dynamics of Cellular Automata* (Wuensche and Lesser 1992), Kauffman's (1993) *The Origins of Order* (Kauffman 1993), and Wuensche's (2016) *Exploring Discrete Dynamics* 2nd Ed (Wuensche 2016)

networks (RBN) (Fig. 15) just in time to make the cover of Kauffman's seminal book (Kauffman 1993) *The Origins of Order* (Fig. 3). The RBN algorithm was later generalized for "discrete dynamical networks" (DDN) described in *Exploring Discrete Dynamics* (Wuensche 2016). The algorithms, implemented in the software DDLab (Wuensche 1993), compute pre-images directly, and basins of attraction are drawn automatically following flexible graphic conventions. There is also an exhaustive "random map" algorithm limited to small systems and a statistical method for dealing with large systems. A more general algorithm can apply to a less general system (MAP → DDN → RBN → CA) for a reality check. The idea of subtrees, basins of attraction, and the entire "basin of attraction field" imposed on state-space is set out in Fig. 4.

The dynamical systems considered in this chapter, whether CA, RBN, or DDN, comprise a finite set of n elements with discrete values v, connected by directed links – the wiring scheme. Each element updates its value synchronously, in discrete time-steps, according to a logical rule applied to its k inputs or a lookup table giving the output of v^k possible input patterns. CA form a special subset with a universal rule and a regular lattice with periodic boundaries, created by wiring from a homogeneous local neighborhood, an architecture that can support emergent complex structure, interacting gliders, glider guns, and universal computation (Conway 1982; Gomez-Soto



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 4 The idea of subtrees, basins of attraction, and the entire "basin of attraction field" imposed on state-space by a discrete dynamical network

and Wuensche 2015; Wuensche 1994a, 1999; Wuensche and Adamatzky 2006). Langton (Langton 1990) has aptly described CA as "a discretized artificial universe with its own local physics."

Classical RBN (Kauffman 1969) have binary values and homogeneous k, but "random" rules and wiring, applied in modeling gene regulatory networks. DDN provide a further generalization allowing values greater than binary and heterogeneous k, giving insights into content-addressable memory and learning (Wuensche 1997). There are countless variations, intermediate architectures, and hybrid systems, between CA and DDN. These systems can also be seen as instances of "random maps with out-degree one" (MAP) (Wuensche 1997, 2016), a list of "exhaustive pairs" where each state in state-space is assigned a random successor, possibly with some bias. All these systems reorganize state-space into basins of attraction.

Running a CA, RBN, or DDN backward in time to trace all possible branching ancestors opens up new perspectives on dynamics. A forward "trajectory" from some initial state can be placed in the context of the "basin of attraction field" which sums up the flow in state-space leading to attractors. The earliest reference I have found to the concept is Ross Ashby's "kinematic map" (Ashby 1956).

For a binary network size n, an example of one of its states B might be 1010 . . . 0110. *State-space* is made up of all $S = 2^n$ states ($S = v^n$ for multi-value) – the space of all possible bitstrings or patterns.

Part of a *trajectory* in state-space, where C is a successor of B and A is a *pre-image* of B, according to the dynamics of the network.

The state B may have other pre-images besides A; the total number is the *in-degree*. The pre-image states may have their own pre-images or none. States without pre-images are known as *garden-of-Eden* or *leaf* states.

Any trajectory must sooner or later encounter a state that occurred previously – it has entered an attractor cycle. The trajectory leading to the attractor is a *transient*. The period of the attractor is the number of states in its cycle, which may be just one – a point attractor.

Take a state on the attractor, find its pre-images (excluding the pre-image on the attractor). Now find the pre-images of each pre-image, and so on, until all leaf states are reached. The graph of linked states is a *transient tree* rooted on the attractor state. Part of the transient tree is a subtree defined by its root.

Construct each transient tree (if any) from each attractor state. The complete graph is the *basin of attraction*. Some basins of attraction have no transient trees, just the bare "attractor."

Now find every attractor cycle in state-space and construct its basin of attraction. This is the *basin of attraction field* containing all 2^n states in state-space but now linked according to the dynamics of the network. Each discrete dynamical network imposes a particular basin of attraction field on state-space.

The term "basins of attraction" is borrowed from continuous dynamical systems, where attractors partition phase-space. Continuous and discrete dynamics share analogous concepts – fixed points, limit cycles, and sensitivity to initial conditions. The separatrix between basins has some affinity to unreachable (garden-of-Eden) leaf states. The spread of a local patch of transients measured by the Lyapunov exponent has its analog in the degree of convergence or bushiness of subtrees. However, there are also notable differences. For example, in discrete systems trajectories are able to merge outside the attractor, so a sub-partition or sub-category is made by the root of each subtree, as well as by attractors.

The various parameters and measures of basins of attraction in discrete dynamics are summarized in the remainder of this chapter. (This review is based on the author's prior publications (Wuensche and Lesser 1992) to (Wuensche 1993) and especially (Wuensche 2010)) together with some insights and applications, firstly for CA and then for RBN/DDN.

Basins of Attraction in CA

Notions of order, complexity, and chaos, evident in the space-time patterns of single trajectories, either subjectively (Fig. 6) or by the variability of input entropy (Figs. 7 and 10), relate to the topology of basins of attraction (Fig. 5). For order, subtrees and attractors are short and bushy. For chaos, subtrees and attractors are long and sparsely branching (Fig. 12). It follows that leaf density for order is high because each forward time-step abandons many states in the past, and unreachable by further forward dynamics – for chaos the opposite is true, with very few states abandoned.

This general law of convergence in the dynamical flow applies for DDN as well as CA, but for CA it can be predicted from the rule itself by its Z-parameter (Fig. 8), the probability that the next unknown cell in a pre-image can be derived unambiguously by the CA reverse algorithm (Wuensche and Lesser 1992; Wuensche 1994a, 1999). As Z is tuned from 0 to 1, dynamics shift from order to chaos (Fig. 8), with transient/attractor length (Fig. 5), leaf density (Fig. 9), and the in-degree frequency histogram (Wuensche 1999, 2016) providing measures of convergence (Fig. 10).

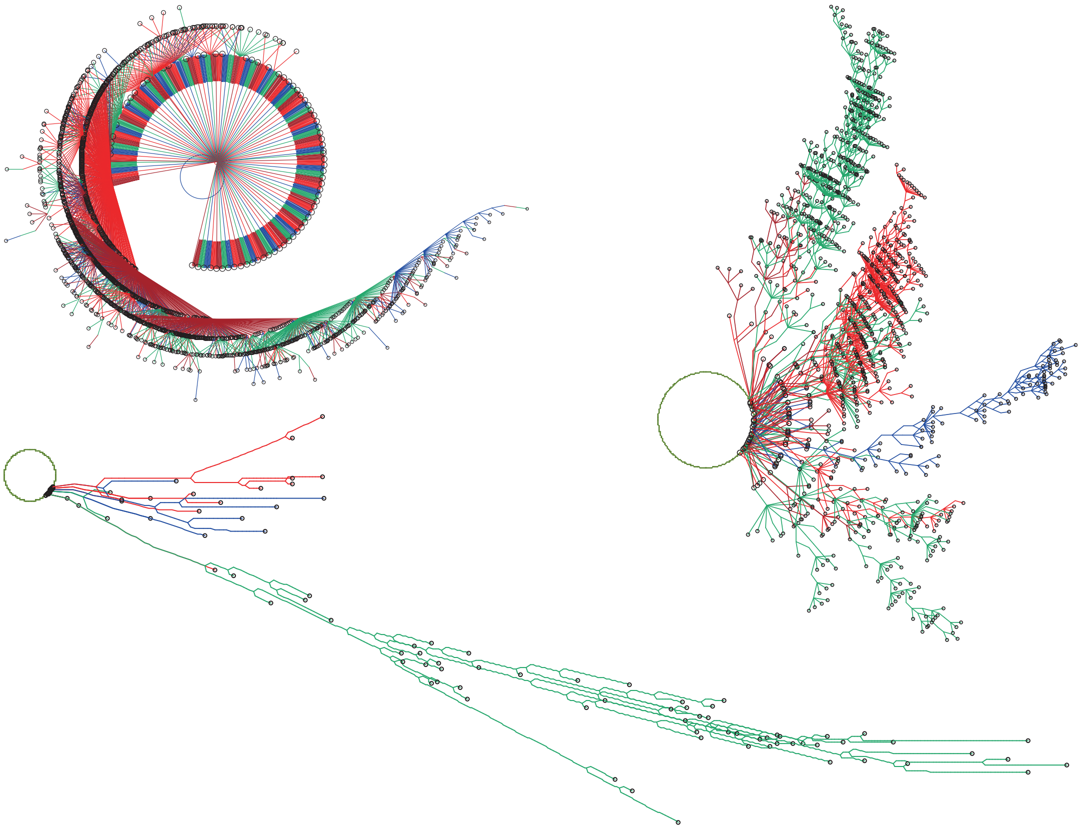
CA Rotational Symmetry

CA with periodic boundary conditions, a circular array in 1D (or a torus in 2D), impose restrictions and symmetries on dynamical behavior and thus on basins of attraction. The "rotational symmetry" is the maximum number of repeating segments s into which the ring can be divided. The size of a repeating segment g is the minimum number of cells through which the circular array can be rotated and still appear identical. The array size $n = s \times g$. For uniform states (i.e., 000000...) $s = n$ and $g = 1$. If n is prime, for any nonuniform state $s = 1$ and $g = n$.

It was shown in (Wuensche and Lesser 1992) that s cannot decrease, may only increase in a transient, and must remain constant on the attractor. So uniform states must occur later in time than any other state – close to or on the attractor, followed by states consisting of repeating pairs (i.e., 010101... where $g = 2$), repeating triplets, and so on. It follows that each state is part of a set of g equivalent states, which make equivalent subtrees and basins of attraction (Wuensche and Lesser 1992; Wuensche 2016, 1993). This allows the automatic regeneration of subtrees once a prototype subtree has been computed and the "compression" of basins – showing just the nonequivalent prototypes, (Fig. 1).

CA Equivalence Classes

Binary CA rules fall into equivalence classes (Walker and Ashby 1966; Wuensche and Lesser 1992) consisting of a maximum of four rules, whereby every rule R can be transformed into its "negative" R_n, its "reflection" R_r, and its "negative/reflection" R_{nr}. Rules in an equivalence class have equivalent dynamics, thus basins of attraction. For example, the 256 $k3$ "elementary rules" fall into 88 equivalence classes whose description



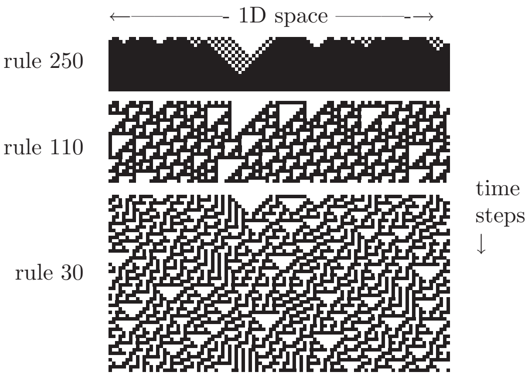
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 5 Three basins of attraction with contrasting topology, $n = 15$, $k = 3$, for CA rules 250, 110, and 30. One complete set of equivalent trees is shown in each case, and just the nodes of unreachable leaf states. The topology varies from very bushy to sparsely branching, with measures such as leaf density, transient length, and in-degree distribution predicted by the rule's *Z*-parameter

suffices to characterize rule-space, and there is a further collapse to 48 "rule clusters" by a complimentary transformation (Fig. 11). Equivalence classes can be combined with their compliments to make "rule clusters" which share many measures and properties (Wuensche and Lesser 1992), including the *Z*-parameter, leaf density, and Derrida plot. Likewise, the 64 *k*5 totalistic rules fall into 36 equivalence classes.

CA Glider Interaction and Basins of Attraction

Of exceptional interest in the study of CA is the phenomenon of complex dynamics. Self-organization and emergence of stable and mobile interacting particles, gliders, and glider guns enables universal computation at the "edge of chaos" (Langton 1990). Notable examples studied for their particle collision logic are the 2D "game-of-life" (Conway 1982), the elementary rule 110 (Cook 2004), and the hexagonal three-value spiral-rule (Wuensche and Adamatzky 2006). More recently discovered is the 2D binary X-rule and its offshoots (Gomez-Soto and Wuensche 2015, 2016).

Here we will simply comment on complex dynamics seen from a basin of attraction perspective (Domain and Gutowitz 1997; Wuensche 1994a), where basin topology and the various measures such as leaf density, in-degree distribution, and the *Z*-parameter are intermediate between order and chaos. Disordered states, before the emergence of particles and their backgrounds, make up leaf states or short dead-end side branches along the length of long transients where particle interactions are progressing. States dominated by particles and their backgrounds are special, a small sub-category of state-space. They constitute the glider interaction phase, making up the main lines of flow within long transients. Gliders in their interaction phase can be regarded as competing sub-attractors. Finally, states made up solely periodic glider interactions, non-interacting gliders, or domains free of gliders must cycle and therefore constitute the relatively short attractors.

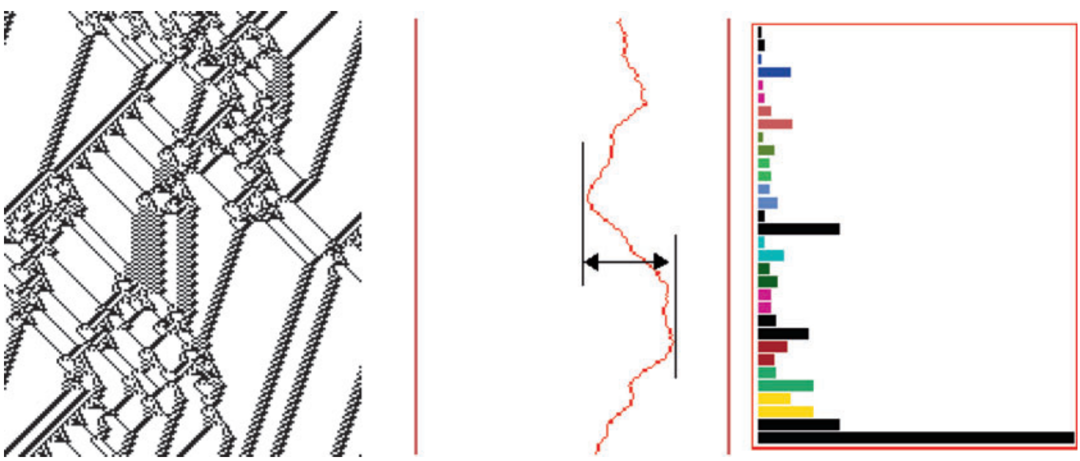


Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 6 1D space-time patterns of the $k = 3$ rules in Fig. 5, characteristic of order, complexity, and chaos. System size $n = 100$ with periodic boundaries. The same random initial state was used in each case. A space-time pattern is just one path through a basin of attraction

Information Hiding within Chaos

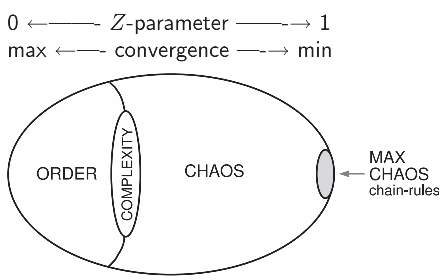
State-space by definition includes every possible piece of information encoded within the size of the CA lattice — including Shakespeare's sonnets, copies of the Mona Lisa, and one's own thumb print, but mostly disorder. A CA rule organizes state-space into basins of attraction where each state has its specific location and where states on the same transient are linked by forward time-steps, so the statement "state $B = A + x$ time-steps" is legitimate. But the reverse "state $A = B - x$" is usually not legitimate because backward trajectories will branch by the in-degree at each backward step, and the correct branch must be selected. More importantly, most states are leaf states without pre-images, or close to the leaves, so for these states "$-x$" time-steps would not exist.

In-degree, convergence in the dynamical flow, can be predicted from the CA rule itself by its



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 7 *Left:* The space-time patterns of a 1D complex CA, $n = 150$ about 200 time-steps. *Right:* A snapshot of the input frequency histogram measured over a moving window of 10 time-steps. *Center:* The changing entropy of the histogram, its variability providing a nonsubjective measure to discriminate between ordered, complex, and chaotic rules automatically. High variability implies complex dynamics. This measure is used to automatically categorize rule-space (Wuensche 1999, 2016) (Fig. 10)

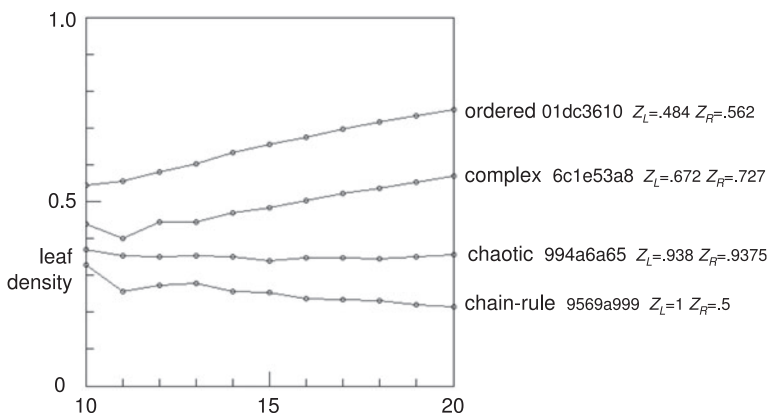
Z-parameter, the probability that the next unknown cell in a pre-image can be derived unambiguously by the CA reverse algorithm (Wuensche and Lesser 1992; Wuensche 1994a, 1999). This is computed in two directions, Z_{left} and Z_{right}, with the higher value taken as *Z*. As *Z* is tuned from 0 to 1, dynamics shift from order to chaos (Fig. 8), with leaf density, a good measure of convergence, decreasing (Figs. 5 and 9). As the system size increases, convergence increases for ordered rules, at a slower rate for complex rules, and remains steady for chaotic rules which make up most of rule-space (Fig. 10). However, there is a class of maximally chaotic "chain" rules where Z_{left} XOR Z_{right} equals 1, where convergence and leaf density decrease with system size *n* (Fig. 9). As *n* increases, in-degrees ≥2, and leaf density, become increasingly rare (Fig. 12) and vanishingly small in the limit. For large *n*, for practical purposes, transients are made up of long chains of states without branches, so it becomes possible to link two states separated in time, both forward and backward. Figure 13 describes how information can be encrypted and decrypted, in this example for an eight-value (eight-color) CA. About the square root of binary rule-space is made up of chain rules, which can be constructed at random to provide a huge number of encryption keys.



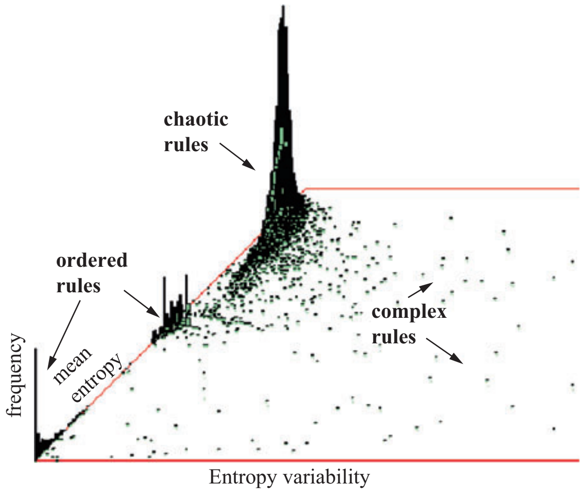
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 8 A view of CA rule-space, after Langton (Langton 1990). Tuning the *Z*-parameter from 0 to 1 shifts the dynamics from maximum to minimum convergence, from order to chaos, traversing a phase transition where complexity lurks. The chain-rules on the right are maximally chaotic and have the very least convergence, decreasing with system size, making them suitable for dynamical encryption.

Memory and Learning

The RBN basin of attraction field (Fig. 15) reveals that content-addressable memory is present in discrete dynamical networks and shows its exact composition, where the root of each subtree (as well as each attractor) categorizes all the states that flow into it, so if the root state is a trigger in some other system, all the states in the subtree could in principle be recognized as belonging to a particular conceptual entity. This notion of memory far from equilibrium (Wuensche 1994b, 1996)

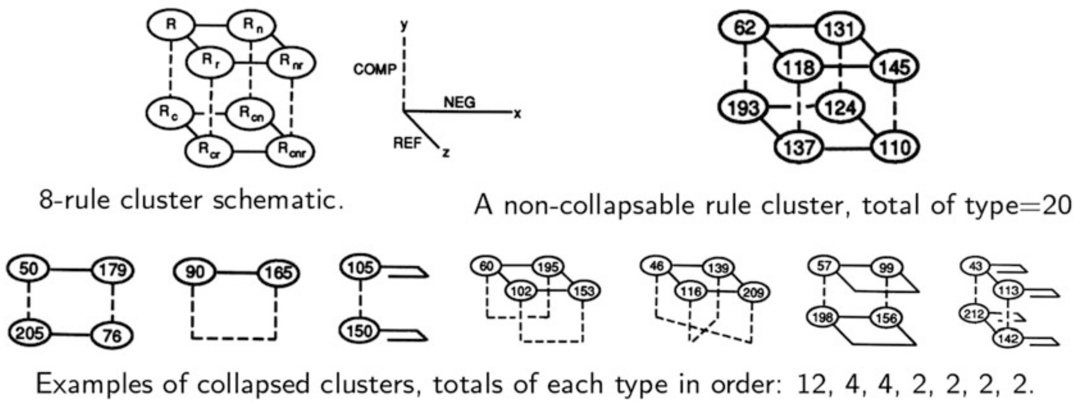


Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 9 Leaf (garden-of-Eden) density plotted against system size *n*, for four typical CA rules, reflecting convergence which is predicted by the *Z*-parameter. Only the maximally chaotic chain-rules show a decrease. The measures are for the basin of attraction field, so for the entire state-space. $k = 5$, $n = 10$–20



	Order	Complexity	Chaos
Mean entropy	Low	Medium	High
Entropy variability	Low	High	Low

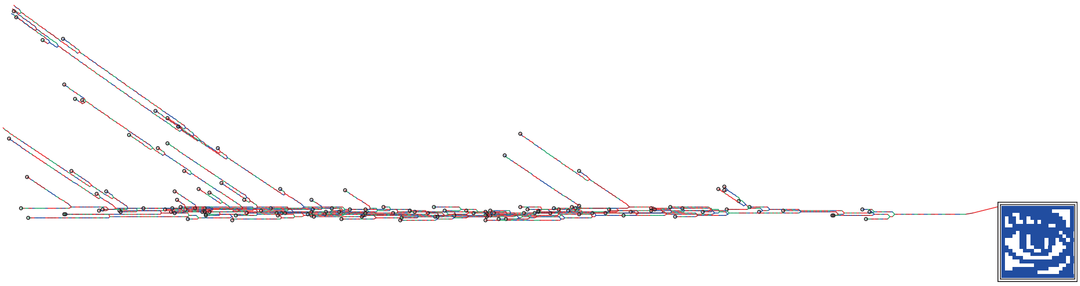
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 10 Scatterplot of a sample of 15,800 2D hexagonal CA rules ($v = 3$, $k = 6$), plotting mean entropy against entropy variability (Wuensche 1999, 2016), which classifies rules between ordered, complex, and chaotic. The vertical axis shows the frequency of rules at positions on the plot – most are chaotic. The plot automatically classifies rule-space as mentioned in the figure



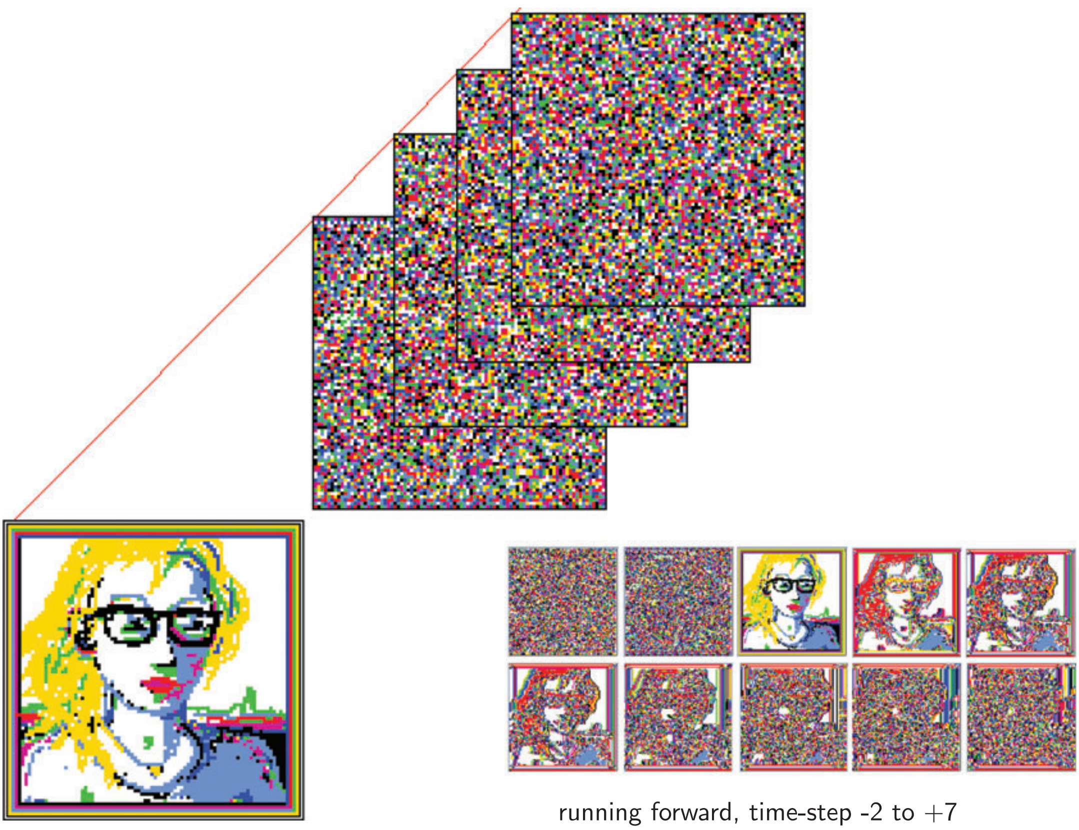
Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 11 Graphical representation of rule clusters of the $v2k3$ "elementary" rules and examples, taken from (Wuensche and Lesser 1992), where it is shown that the 256 rules in rule-space break down into 88 equivalence classes and 48 clusters. The rule cluster is depicted as two complimentary sets of four equivalent rules at the corners of a box – with negative, reflection, and complimentary transformation links on the x, y, z edges, but these edges may also collapse due to identities between a rule and its transformation

extends Hopfield's (Hopield 1982) and other classical concepts of memory in artificial neural networks, which rely just on attractors.

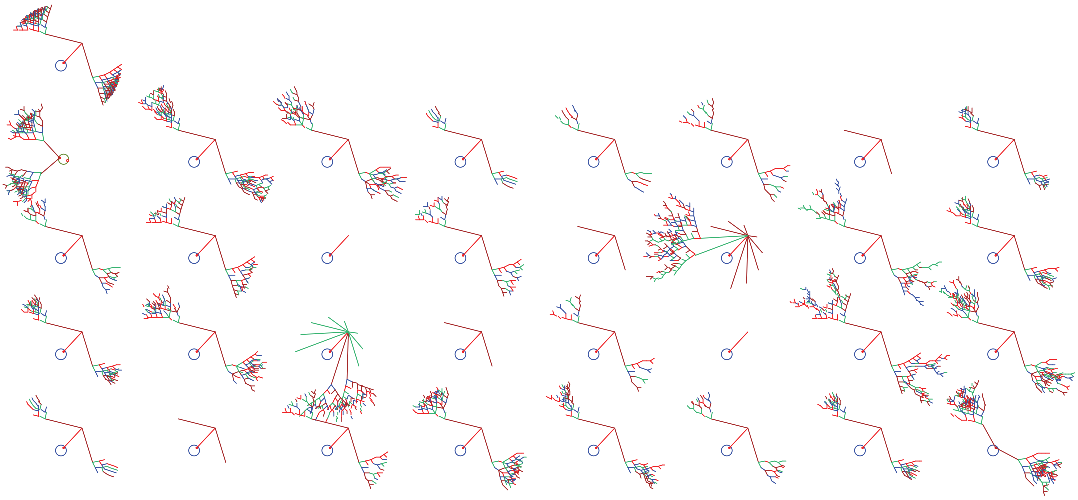
As the dynamics descend toward the attractor, a hierarchy of sub-categories unfolds. Learning in this context is a process of adapting the rules and connections in the network, to modify subcategories for the required behavior – modifying the fine structure of subtrees and basins of attraction. Classical CA are not ideal systems to



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 12 A subtree of a chain-rule 1D CA $n = 400$. The root state (the eye) is shown in 2D (20×20). Backward iteration was stopped after 500 reverse time-steps. The subtree has 4270 states. The density of both leaf states and states that branch is very low (about 0.03) – where maximum branching equals 2



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 13 *Left*: A 1D pattern is displayed in 2D ($n = 7744$, 88×88). The "portrait" was drawn with the drawing function in DDLab. With a $v = 8$, $k = 4$ chain-rule constructed at random, and the portrait as the root state, a subtree was generated with the CA reverse algorithm, set to stop after four backward time-steps. The state reached is the encryption. To decrypt, run forward by the same number of time-steps. *Right*: Starting from the encrypted state, the CA was run forward to recover the original image. This figure shows time-steps from −2 to +7 to illustrate how the image was scrambled both before and after time-step 0



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 14 Mutant basins of attraction of the $v = 2$, $k = 3$, rule 60 ($n = 8$, seed all 0 s). *Top left*: The original rule, where all states fall into just one very regular basin. The rule was first transformed to its equivalent $k = 5$ rule (f00ff00f in hex), with 32 bits in its rule table. All 32 one-bit mutant basins are shown. If the rule is the genotype, the basin of attraction can be seen as the phenotype

implement these subtle changes, restricted as they are to a universal rule and local neighborhood, a requirement for emergent structure, but which severely limits the flexibility to categorize. Moreover, CA dynamics have symmetries and hierarchies resulting from their periodic boundaries (Wuensche and Lesser 1992). Nevertheless, CA can be shown to have a degree of stability in behavior when mutating bits in the rule table – with some bits more sensitive than others. The rule can be regarded as the genotype and basins of attraction as the phenotype (Wuensche and Lesser 1992). Figure 14 shows CA mutant basins of attraction.

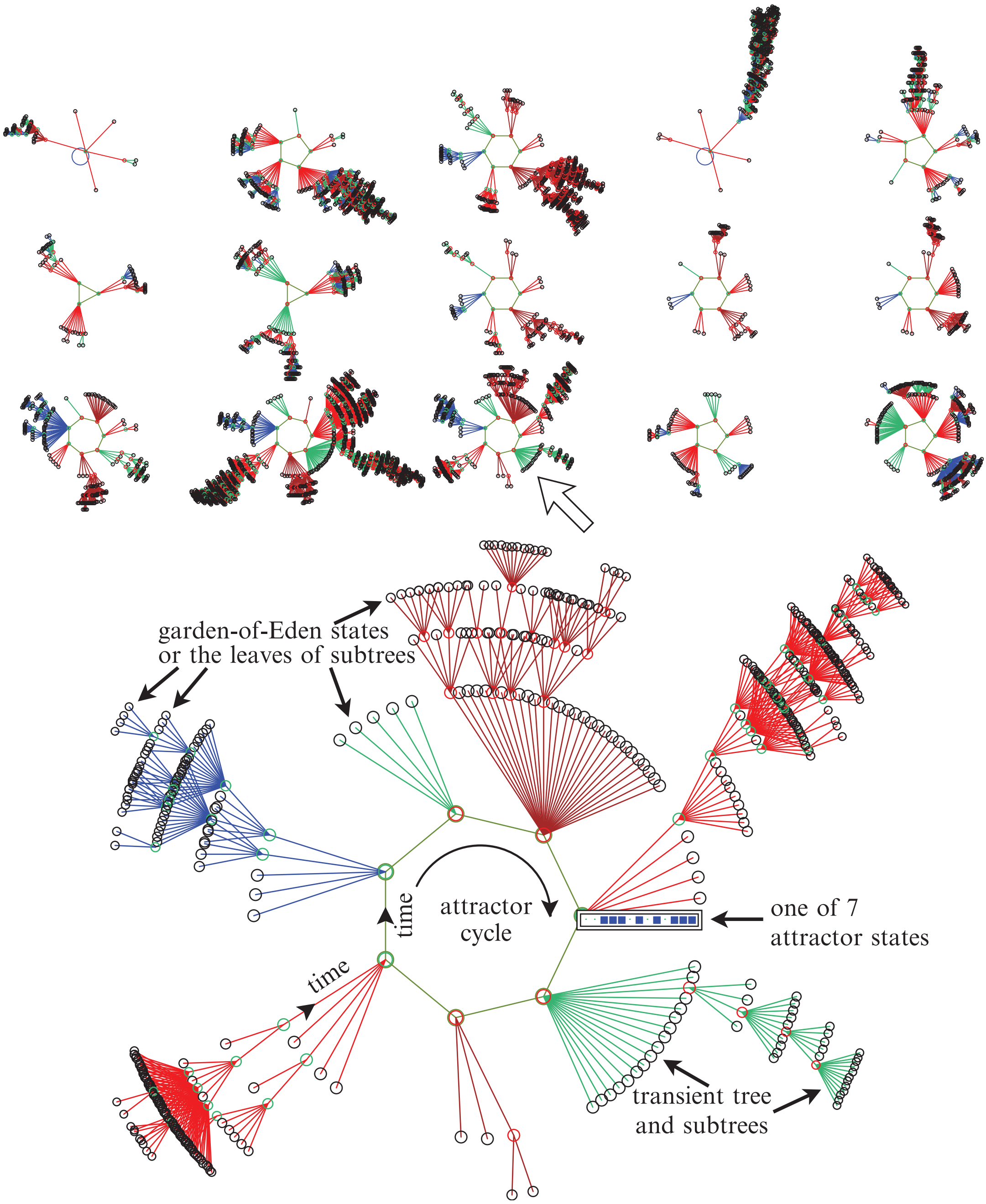
With RBN and DDN there is greater freedom to modify rules and connections than with CA (Fig. 15). Algorithms for learning and forgetting (Wuensche 1994b, 1996, 1997) have been devised, implemented in DDLab. The methods assign pre-images to a target state by correcting mismatches between the target and the actual state, by flipping specific bits in rules or by moving connections. Among the side effects, generalization is evident, and transient trees are sometimes transplanted along with the reassigned pre-image.

Modeling Neural Networks

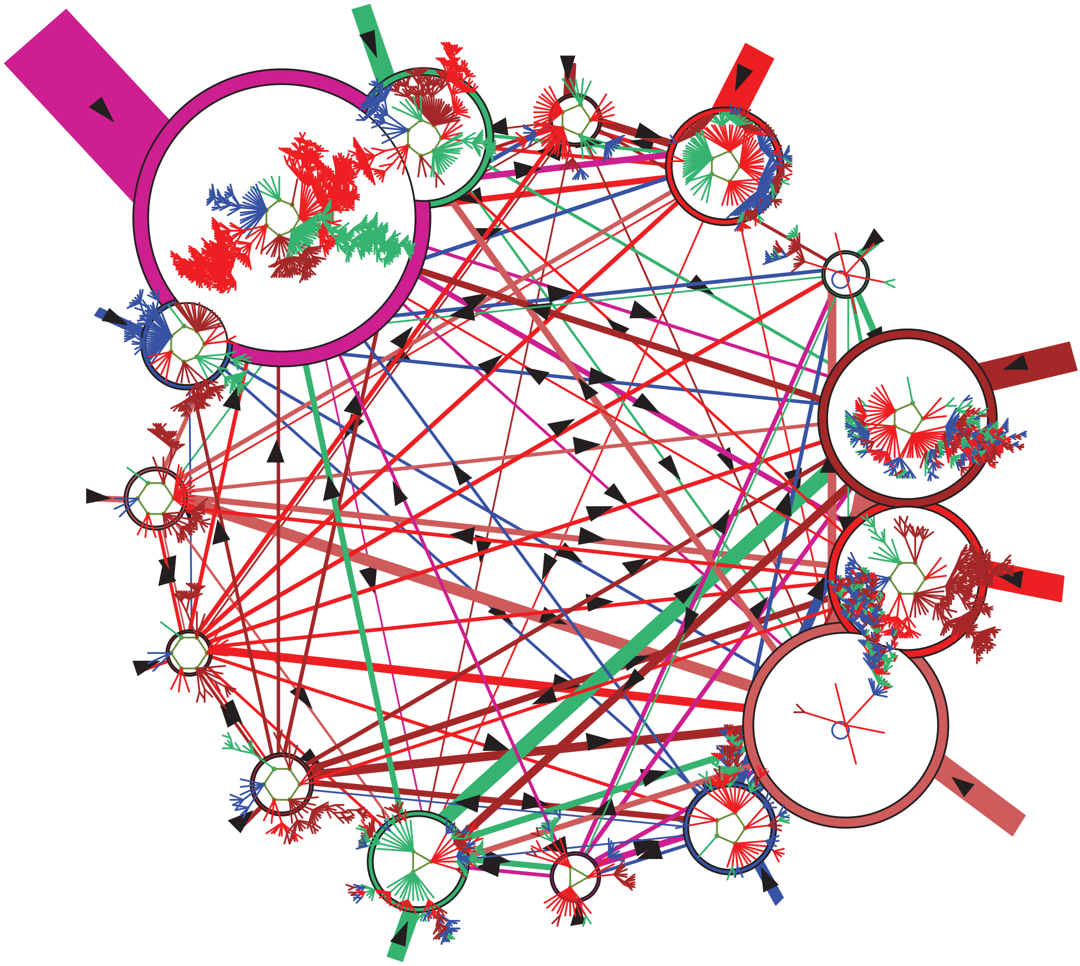
Allowing some conjecture and speculation, what are the implications of the basin of attraction idea on memory and learning in animal brains (Wuensche 1994b, 1996)? The first conjecture, perhaps no longer controversial, is that the brain is a dynamical system (not a computer or Turing machine) composed of interacting subnetworks. Secondly, neural coding is based on distributed patterns of activation in neural subnetworks (not the frequency of firing of single neurons) where firing is synchronized by many possible mechanisms: phase locking, interneurons, gap junctions, membrane nanotubes, and ephaptic interactions.

Learnt behavior and memory work by patterns of activation in subnetworks flowing automatically within the subtrees of basins of attraction. Recognition is easy because an initial state is provided. Recall is difficult because an association must be conjured up to initiate the flow within the correct subtree.

At a very basic level, how does a DDN model a semiautonomous patch of neurons in the brain whose activity is synchronized? A network's connections model the subset of neurons connected to a given neuron. The logical rule at a network



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 15 *Top:* The basin of attraction field of a random Boolean network, $k = 3$, $n = 13$. The $2^{13} = 8192$ states in state-space are organized into 15 basins, with attractor periods ranging between 1 and 7 and basin volume between 68 and 2724. *Bottom:* A basin of attraction (arrowed above) which links 604 states, of which 523 are leaf states. The attractor period = 7, and one of the attractor states is shown in detail as a bit pattern. The direction of time is inward and then clockwise at the attractor



Basins of Attraction of Cellular Automata and Discrete Dynamical Networks, Fig. 16 The jump graph (of the same RBN as in Fig. 15) shows the probability of jumping between basins due to single bit-flips to attractor states. Nodes representing basins are scaled according the number of states in the basin (basin volume). Links are scaled according to both basin volume and the jump probability. Arrows indicate the direction of jumps. Short stubs are self-jumps; more jumps return to their parent basin than expected by chance, indicating a degree of stability. The relevant basin of attraction is drawn inside each node

element, which could be replaced by the equivalent treelike combinatorial circuit, models the logic performed by the synaptic microcircuitry of a neuron's dendritic tree, determining whether or not it will fire at the next time-step. This is far more complex than the threshold function in artificial neural networks. Learning involves changes in the dendritic tree or, more radically, axons reaching out to connect (or disconnect) neurons outside the present subset.

Modeling Genetic Regulatory Networks

The various cell types of multicellular organisms, muscle, brain, skin, liver, and so on (about 210 in humans), have the same DNA so the same set of genes. The different types result from different patterns of gene expression. But how do the patterns maintain their identity? How does the cell remember what it is supposed to be?

It is well known in biology that there is a genetic regulatory network, where genes regulate each other's activity with regulatory proteins

(Somogyi and Sniegoski 1996). A cell type depends on its particular subset of active genes, where the gene expression pattern needs to be stable but also adaptable. More controversial to cell biologists less exposed to complex systems is Kauffman's classic idea (Kauffman 1969, 1993; Wuensche 1998) that the genetic regulatory network is a dynamical system where cell types are attractors which can be modeled with the RBN or DDN basin of attraction field. However, this approach has tremendous explanatory power, and it is difficult to see a plausible alternative.

Kauffman's model demonstrates that evolution has arrived at a delicate balance between order and chaos, between stability and adaptability, but leaning toward convergent flow and order (Harris et al. 2002; Kauffman 1993). The stability of attractors to perturbation can be analyzed by the jump graph (Fig. 16) which shows the probability of jumping between basins of attraction due to single bit-flips (or value-flips) to attractor states (Wuensche 2004, 2016). These methods are implemented in DDLab and generalized for DDN where the value range, *v*, can be greater than 2 (binary), so a gene can be fractionally on as well as simply on/off.

A present challenge in the model, the inverse problem, is to infer the network architecture from information on space-time patterns and apply this to infer the real genetic regulatory network from the dynamics of observed gene expression (Harris et al. 2002).

Future Directions

This chapter has reviewed a variety of discrete dynamical networks where knowledge of the structure of their basins of attraction provides insights and applications: in complex cellular automata particle dynamics and self-organization, in maximally chaotic cellular automata where information can be hidden and recovered from a stream of chaos, and in random Boolean and multi-value networks that are applied to model neural and genetic networks in biology. Many avenues of inquiry remain – whatever the discrete dynamical system, it is worthwhile to think about it from the basin of attraction perspective.

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