Basins of Attraction in Cellular Automata

Order–Complexity–Chaos in Small Universes

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1992 saw the publication of *The Global Dynamics of Cellular Automata* [1]. (See Stuart Kauffman's review in this issue, page 47.) This book developed the notion of basins of attraction in one-dimensional binary cellular automata (CA), together with an "atlas" for two entire categories of rule-space. CA had usually been represented just by their space-time patterns, typical trajectories from various initial states. A state is the pattern of 0s and 1s at a given time-step. If a trajectory is an example of local dynamics, then all possible merging trajectories sum up the system's global dynamics.

CA are very simple discrete dynamical networks where each "cell" simultaneously updates its value as a logical function of its own value and its close neighbors. In one dimension the cells form a ring. The possible logics that can be applied, rule-space, give rise to a range of recognizable behavior in space-time patterns, from order to "chaos," and also to complex dynamics at the transition, according to various static rule parameters [1,2]. Complex dynamics, where interacting particles or gliders take over, make CA especially interesting as an example of the emergence of complex spatial pattern in the simplest possible system. In this case behavior can be described at a higher level, by a catalog of gliders and their interactions, quite apart from the underlying "physics."

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To achieve the global perspective, I devised a general method for running CA backwards in time to compute a state's predecessors with a direct reverse algorithm. So the predecessors of predecessors, an so on, can be computed, revealing the complete subtree including the "leaves," states without predecessors, the so-called "garden-of-Eden" states.

Trajectories must lead to attractors in a finite CA, so a basin of attraction is composed of merging trajectories, trees, rooted on the states making up the attractor cycle with a period of one or more. State-space is organized by the "physics" underlying the dynamic behavior into a number of these basins of attraction, making up the basin of attraction field.

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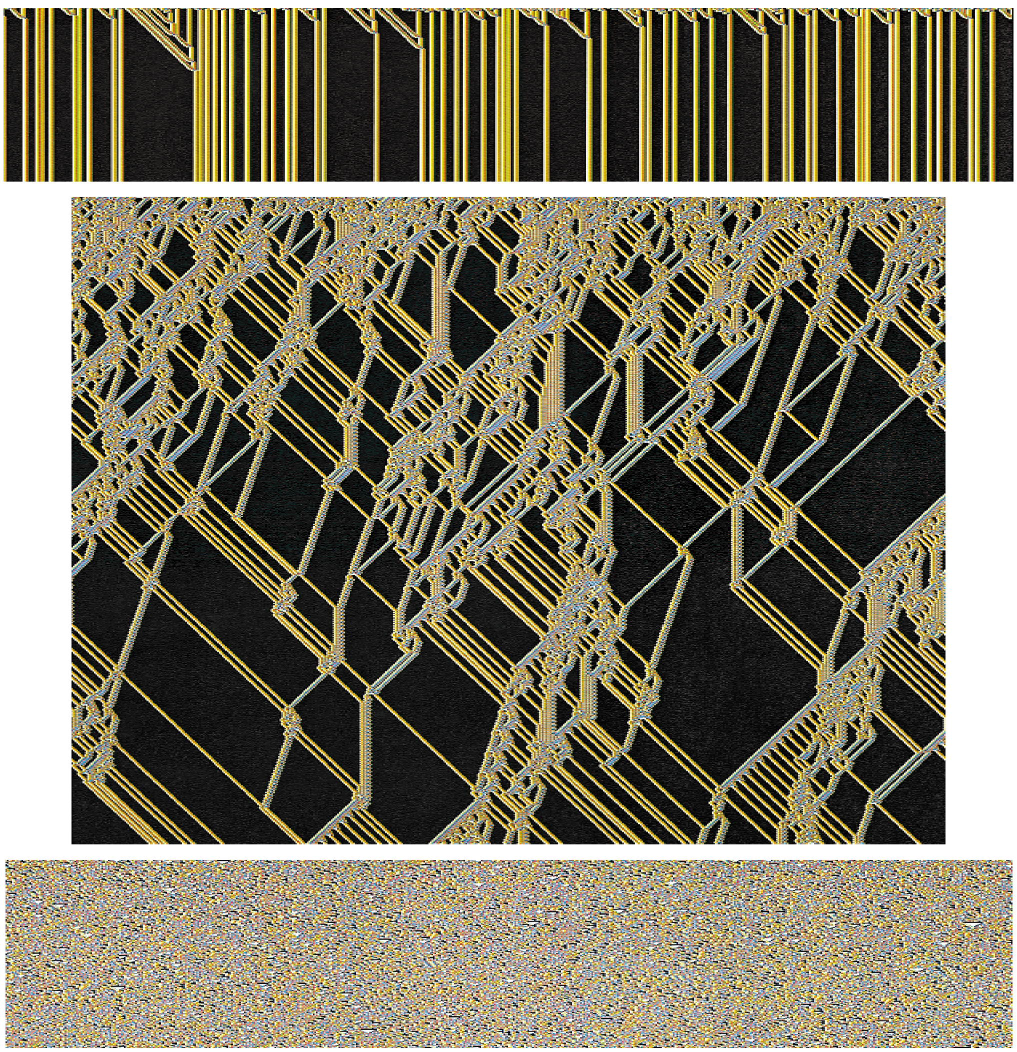
The earliest reference I have found to this, as just a concept, is Ross Ashby's "kinematic map" [3]. The terminology I use is borrowed in part from continuous dynamical systems, where attractors partition phase-space. There are many analogous properties such as "chaos" with sensitivity to initial conditions, but also notable differences. For example, in these discrete systems trajectories are able to merge outside the attractor, so the root state of each subtree makes a subpartition, as well as the attractors themselves.

This is important in understanding categorization, or memory, in discrete networks, where Boolean networks are applied to model both neural networks [4] and genetic regulatory networks [5,6]. Boolean networks are a generalization of CA where the connections and logic may be different at each cell. However, the same basin of attraction concepts apply. Following the CA reverse algorithm, I devised a Boolean network reverse algorithm so their basins of attraction can also be computed [4].

For CA, with both the local and global perspective available, comparisons can be made between measures on space-time patterns and measures on the basins of attraction, and also with static rule parameters [7]. Order-complexity-chaos in space-time patterns is recognizable to the subjective eye, but there are also objective measures that closely correlate, such as the variance of input-entropy.

Examples of measures on basins of attraction are the number of attractors, attractor periods, and the length of transients. It turns out that a key measure is the typical branchiness or bushiness of subtrees, whether states have many predecessors or few (the in-degree), relative to system size. This is measured by the fraction of garden-of-

FIGURE 1



Space-time patterns from a random initial state, for ordered, complex, and chaotic rules, $k = 5$, $n = 900$. For the same rules, the basin of attraction fields are shown in Figure 3 and subtrees in Figure 4. Top: ordered rule 0ldc3610, center; complex rule 6cle53a8, bottom: chaotic rule 994a6a65.

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Eden states (those with no predecessors) and more precisely by the frequency distribution of all in-degrees.

Ordered dynamics is highly convergent, with high in-degree, resulting in extremely bushy subtrees with a very high fraction of garden-of-Eden states, short attractor periods, and short transients because many states are used up at each forward iteration.

Chaotic dynamics has low convergence, low in-degree, with a low fraction of garden-of-Eden states, long attractor periods, and long transients because few states are used up at each forward iteration.

Complex dynamics has intermediate convergence, with the frequency of in-degrees following a power law distribution.

For an in-depth study, see the research article in *Complexity* [7]

THE IMAGES

These images are taken from an art show, "Objective Wonder: Data as Art," held at the University of Arizona in March 1999. Our exhibit was titled "Complexity in Small Universes," created in collaboration with Chris Langton, and focused on the idea that complexity is bounded by order and disorder in one-dimensional binary CA. A CA can be regarded as a small formal universe with a very simple physics. More images from the exhibit can be seen at www.santafe.edu/˜wuensch/Exh2/Exh3.html/.

The images were created with Discrete Dynamics Lab (DDLab), my software package for studying discrete dynamical networks, from cellular automata to Boolean networks, available at www.santafe.edu/˜wuensch/ddlab.html/, and reviewed in *Complexity* Vol. 3/No. 1. Further DDLab images with explanatory notes are at www.santafe.edu/˜wuensch/gallery/ddlab_gallery.html/.

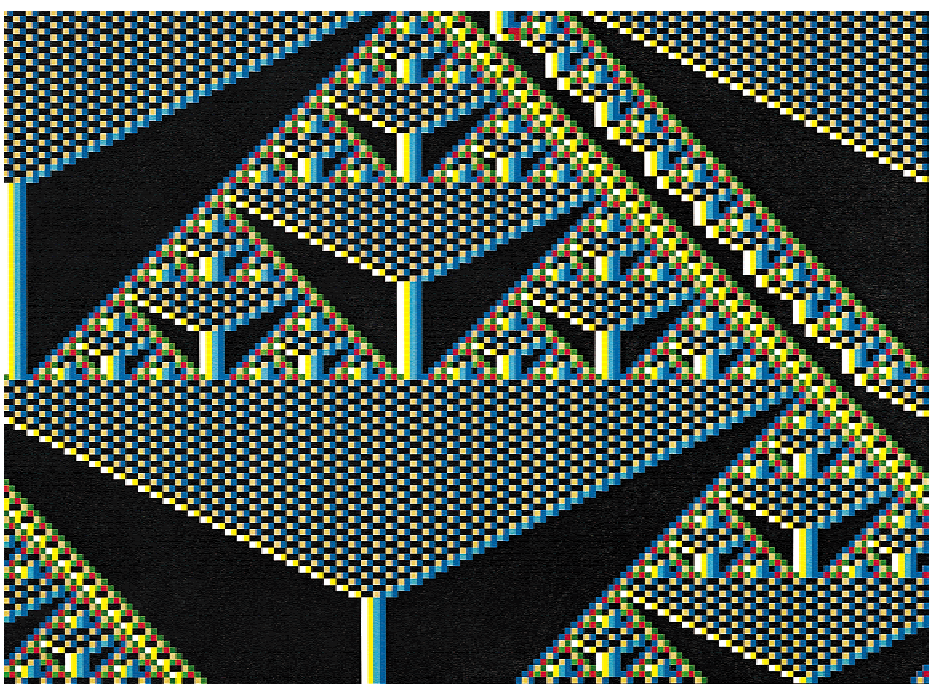
The images are labeled with the neighborhood size, *k,* the rule number (in hex for *k5,* in decimal for *k3*), and the system size *n*, so they may all be reproduced with DDLab.

Figure 1 shows space-time patterns from a random initial state, for examples of ordered, complex, and chaotic CA. Space extends horizontally, and successive states in time are plotted vertically one below the other. Figure 2 shows another example of a complex rule. In this presentation a cell's color depends on its neighborhood at the previous time-step rather than its value, 0 or 1, and may be "filtered" to show up the interacting structure more clearly.

The remaining figures show examples of basin of attraction fields, single basins, and subtrees, for ordered, complex, and chaotic CA. Those in Figures 3 and 4 are for the same rules as in Figure 1.

These are computer-generated graphs representing the "flow" between states, where each unique state, a particular point in time in a space-time pattern (a whole 1D horizontal slice), is represented as a vertex in the graph. For practical reasons of graph size and computational load, the system sizes are much smaller than for the space-time patterns in Figure 1, though for subtrees the size can be much larger than for basin of attraction fields.

FIGURE 2



The space-time pattern of a complex rule (discovered by Wentian Li) showing a fractal structure. n = 150, k = 5, rule c3bce390.

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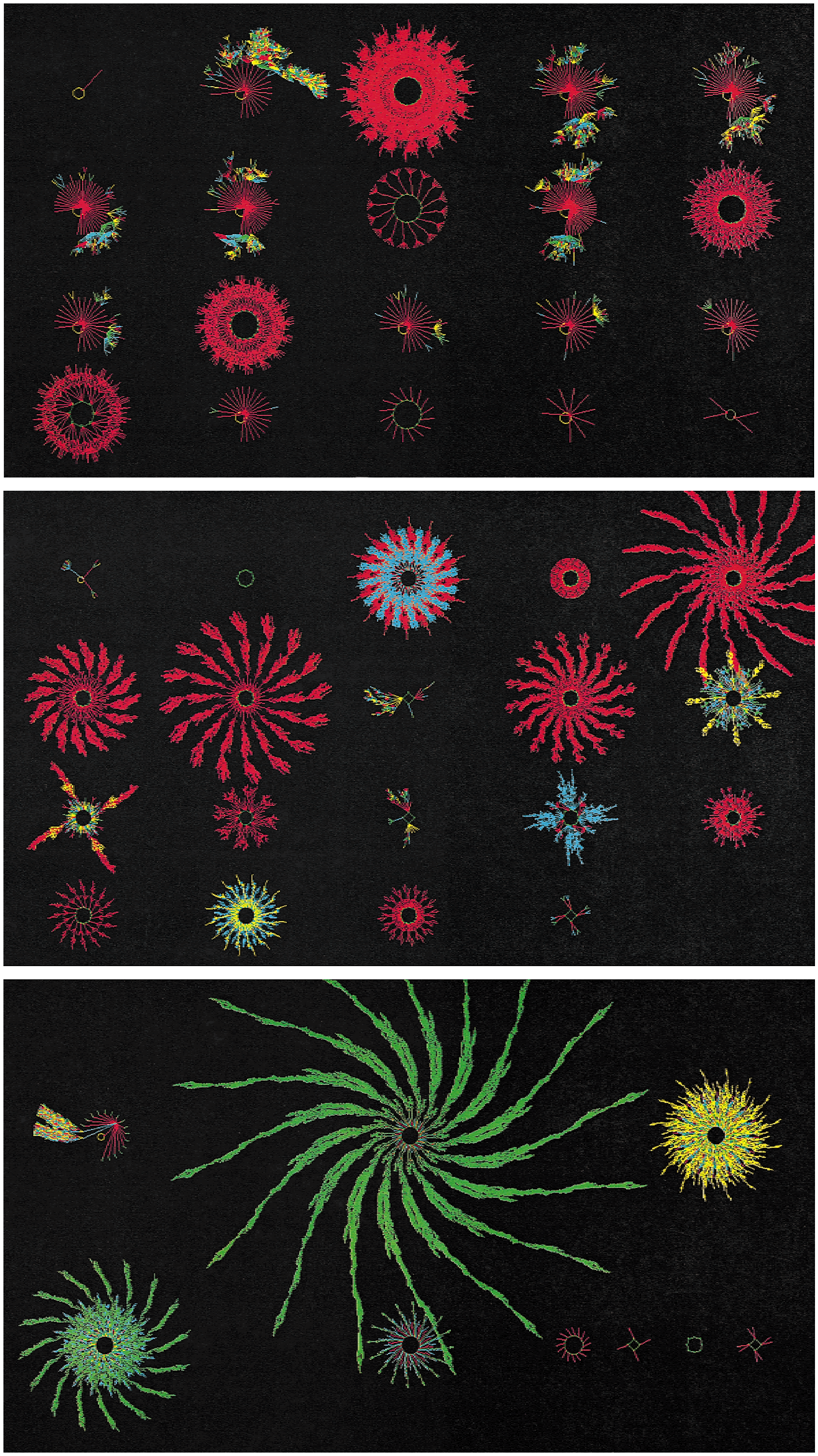
FIGURE 3



The basin of attraction fields of the same ordered, complex, and chaotic rules as in Figure 1 for space-time patterns and Figure 4 for typical subtrees. $n = 16$, $k = 5$. Top: order, center: complexity, bottom: chaos.

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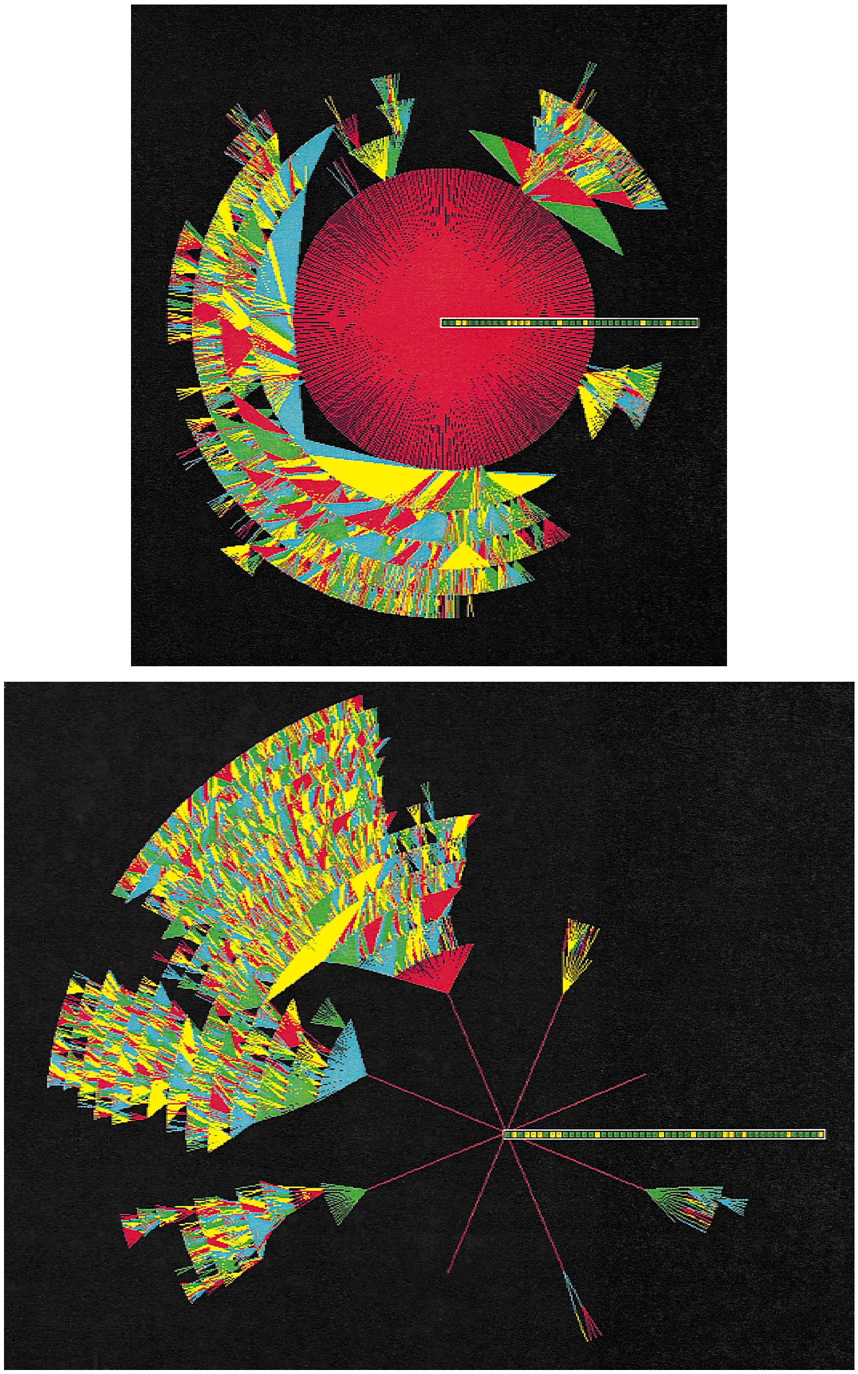
FIGURE 4



Typical subtrees for the same ordered, complex and chaotic rules as in figure 1 for space-time patterns, and figure 3 for basin of attraction fields. n = 40 for the ordered rule (top), n = 50 for the complex rule (bottom), and for the chaotic rule see next page. The subtree root state is shown as a bit pattern.

Each vertex has exactly one outgoing arc because the system is deterministic, but any number of incoming arcs (the in-degree) from its predecessor vertices. Vertices with zero in-degree are the "leaves" of subtrees representing garden-of-Eden states.

The graphs for subtrees consist of arcs converging on a central vertex, the root of the subtree. The graphs for basins of attraction consist of trees rooted on an attractor cycle. If the attractor period is one, the "point attractor" is shown cycling to itself. Arcs between vertices on the attractor are made

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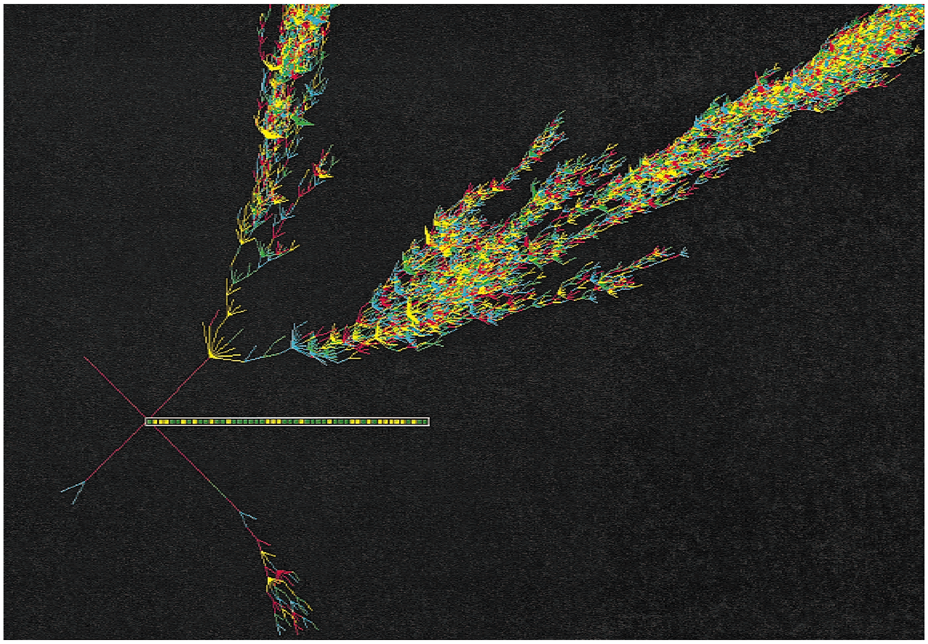
FIGURE 4 Continued



shorter with increasing period to confine the attractor diameter, and in general arcs are made shorter further out from the attractor to contain the graph. The direction of time is inward from garden-of-Eden states toward the subtree root or toward the attractor, and then clockwise around the attractor cycle.

For basin of attraction fields, equivalent basins occur because of various rotation symmetries of the periodic one-dimensional CA [1]. Only one prototype of each equivalent basin is shown. Equivalent subtrees in basins of attraction can also be suppressed.

Colors are assigned (cycling through 4 colors) to successive nonequivalent

FIGURE 5 Continued

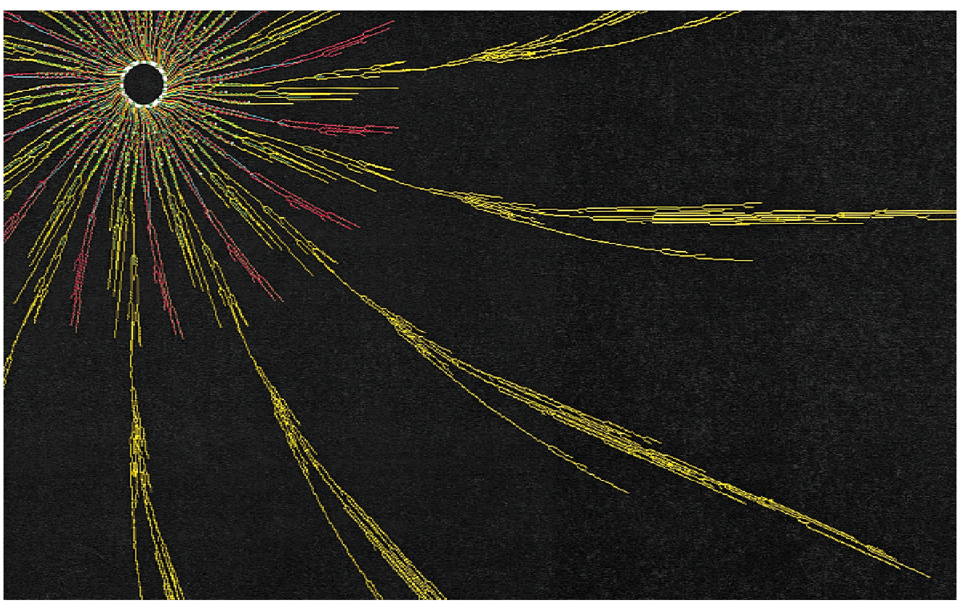


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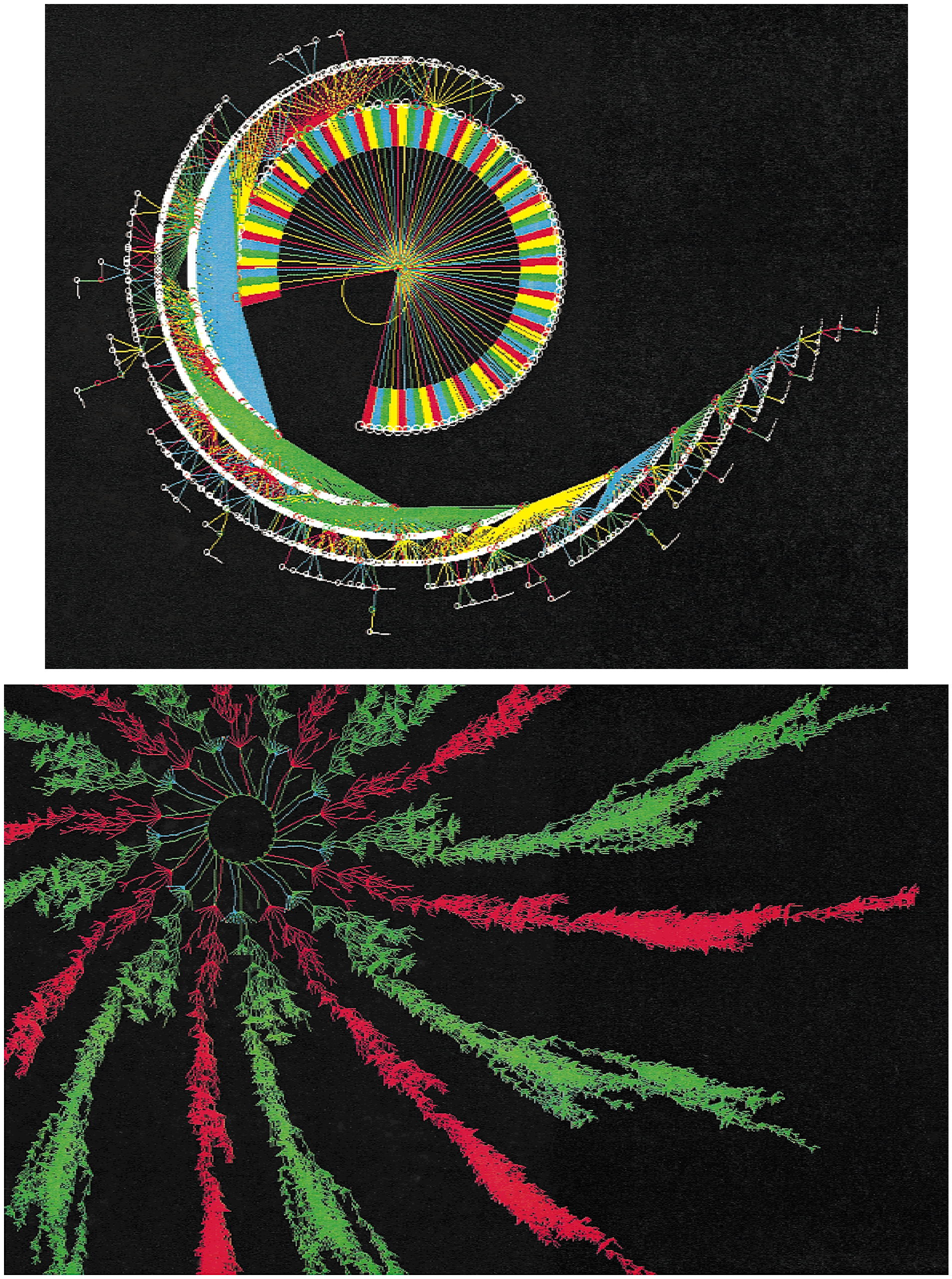
FIGURE 5



Typical single basins of attraction for other examples of ordered, complex, and chaotic rules. Top: $n = 15$, $k = 3$, ordered rule 250. Bottom: $n = 18$, $k = 3$, complex rule 110. Previous page: $n = 15$, $k = 3$, chaotic rule 30.

subtrees rooted on attractor cycles or to the arcs converging onto successive states in subtrees and in basins with very small attractor periods.

These graphic conventions have been devised to give a clear impression of attractor basins, but it should be remembered that the essential information is how states are connected, not the particular appearance of the basin images.

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